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AFOSR 4750

62-4-2

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WIENER-HERMITE FUNCTIONAL EXPANSION IN TURBULENCE WITH THE BURGERS MODEL*

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JUN 19 1963
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*Work of A. S. (in part) and T. I. (in full) supported by Air Force Office of Scientific Research; of W. C. M. (in part) supported by Office of Naval Research.

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This research was supported by the General Physics Division, AFOSR, SRPP under ~~Contract~~/Grant 62-177

Our ultimate aim is to exploit the nearness to Gaussianity of velocity probability distributions in turbulence, by expanding the velocity field function about the Gaussian approximation. Many of the mathematical manifestations of our method are so new, however, that it is hard to obtain physical insight into them. Hence, we have undertaken a preliminary, pilot project of a simplified nature, which uses instead of the Navier-Stokes or MHD equations the Burgers one-dimensional model equation:[1]

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - \nu\,\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

The expansion is in terms of Wiener-Hermite functionals:[2,3,4]

$$u(x,t) = \int K^{(1)}(x-x_1)H^{(1)}(x_1)\,dx_1 + \iint K^{(2)}(x-x_1,x-x_2)H^{(2)}(x_1,x_2)dx_1\,dx_2 + \cdots \tag{2}$$

The $K^{(i)}$, which are implicitly functions of t = time, are ordinary (non-random) functions. Integrations are over $(-\infty,\infty)$. The $H^{(i)}$ are the Wiener-Hermite functionals as defined in Reference 4. They are random functions, whose salient properties are (a) their mutual statistical orthogonality, expressed by such relations as

$$\left\langle H^{(i)}\,H^{(j)}\right\rangle = 0 \text{ if } i \neq j,\ \left\langle H^{(1)}(x_1)\,H^{(1)}(x_2)\right\rangle = \delta(x_1-x_2), \text{ etc.};$$

(b) $H^{(1)}$ is the "ideal random function"[4,5], or derivative of the Wiener function[6,7], whence the first term of (2) is a Gaussian function, and the later terms are in some sense successively higher corrections to Gaussianity; (c) the expansion is rigorously convergent for a broad class of random

functions, hence should converge rapidly whenever the velocity distribution is approximately Gaussian, which is particularly the case in the final stage of turbulence.

If this expression for $u(x,t)$ is substituted into (1), one can obtain from the statistical orthogonality relations an infinite set of equations relating the $K^{(i)}$. When the $K^{(i)}$ are Fourier-transformed according to the relation

$$K^{(i)}(x_1,\cdots x_i) = \frac{1}{(2\pi)^i}\int e^{-i(x_1\chi_1+\cdots+x_i\chi_i)}K^{(i)}(\chi_1,\cdots\chi_i)\, d\chi_1\cdots d\chi_i, \tag{3}$$

these equations become

$$\frac{\partial}{\partial t}K^{(1)}(\chi) + \nu\,\chi^2 K^{(1)}(\chi) + \cdots = 0 \tag{4}$$

$$\frac{\partial}{\partial t}K^{(2)}(\chi_1,\chi_2) + \nu(\chi_1+\chi_2)^2K^{(2)}(\chi_1,\chi_2) - \frac{i}{2}(\chi_1+\chi_2)K^{(1)}(\chi_1)K^{(1)}(\chi_2) + \cdots = 0 \tag{5}$$

$$\begin{aligned}&\frac{\partial}{\partial t}K^{(3)}(\chi_1,\chi_2,\chi_3) + \nu(\chi_1+\chi_2+\chi_3)^2K^{(3)}(\chi_1,\chi_2,\chi_3) - \\ &\frac{i}{3}(\chi_1+\chi_2+\chi_3)[K^{(1)}(\chi_1)K^{(2)}(\chi_2,\chi_3) + K^{(1)}(\chi_2)K^{(2)}(\chi_3,\chi_1) + \\ &K^{(1)}(\chi_3)K^{(2)}(\chi_1,\chi_2)] + \cdots = 0\end{aligned} \tag{6}$$

The dots in equations (4) through (6) stand for an infinite number of terms, involving kernels with higher indices; for example, equation (4) contains a term $-2i\chi(K^{(1)}\cdot K^{(2)})$, where the inner product is defined as

$$(K^{(1)}\cdot K^{(2)}) = \frac{1}{2\pi}\int \overline{K^{(1)}(\chi')}K^{(2)}(\chi,\chi')\,d\chi' \tag{7}$$

Formalizing the assumption that the series (2) converges rapidly, we assign to such inner products (subject to verification) an order of

smallness equal to the sum of the subscripts. Thus, the terms given explicitly in equation (4) are of first order, the omitted term just cited is of third order, and all others are of still higher order. If we retain only the terms written down in equations (4) through (6), we have approximate relations, homogeneous in order of smallness, which determine (subject to initial conditions) each of the first three kernels to lowest order.

As initial conditions we assume

$$K^{(1)}(\chi)\Big|_{t=0} = i\,A\chi^m\, e^{-\ell^2\chi^2} \tag{8}$$

$$K^{(2)}(\chi_1,\chi_2)\Big|_{t=0} = i\,B(\chi_1+\chi_2)^n\, e^{-\ell^2(\chi_1^2+\chi_2^2)} \tag{9}$$

$$K^{(3)}(\chi_1,\chi_2,\chi_3)\Big|_{t=0} = i\,C(\chi_1+\chi_2+\chi_3)^p\, e^{-\ell^2(\chi_1^2+\chi_2^2+\chi_3^2)} \tag{10}$$

where A, B, and C are constants. These conditions are "initial" ones only when considered relative to a stage of decay late enough to be characterizable by a single correlation length ℓ. The exponentials are characteristic forms for the late decay of the diffusion-like equations (4) to (6)[8]. The terms χ^m, $(\chi_1+\chi_2)^n$, $\cdots$ are presumably the leading terms of Taylor expansions, and m, n are assumed integers. One finds that m = 0 leads to $(K^{(2)}:K^{(2)})$ (double inner product) constant in time, which means a non-decaying mean velocity, which is physically untenable. The choice m = 1 leads to $(K^{(2)}:K^{(2)})$ decaying and at the same rate as $(K^{(1)}\cdot K^{(2)})$.

As our starting point we chose m = n = p = 1. We have integrated equations (4) - (6) (we omit the expressions for the K's so obtained, for lack of space) and obtained some simply interpretable results from them:

The omitted terms in the equations of the $K^{(i)}$, when evaluated in terms of these approximate solutions, decay, relatively to those retained, as $t^{-1/2}$ or faster. This guarantees the self-consistency of omitting them. (This result is obtained by simple dimensional analysis of the expressions involved.)

The distribution approaches a Gaussian one in time: the flatness factor of the one-point distribution is, in terms of the $K^{(i)}$, and to the present order of approximation,

$$\Phi \simeq 3 + 24 \frac{(K^{(1)}K^{(1)}K^{(1)} \vdots K^{(3)}) + 2\left((K^{(1)} \cdot K^{(2)}) \cdot (K^{(1)} \cdot K^{(2)})\right)}{(K^{(1)} \cdot K^{(1)})^2} \tag{11}$$

The fraction in the second term, apart from the factor 24, is a measure of the relative deviation from Gaussianity ("R.D.G."), which we denote by ϕ. The present assumptions lead to ϕ decaying as $t^{-1/2}$. (It is important to note that, contrary to what one might think, this is consistent with the equal rates of decay mentioned above for $(K^{(2)}:K^{(2)})$ and $(K^{(1)} \cdot K^{(1)})$.

As a venture in investigating the predictive value of the inherent mathematical structure of the R.D.G. in this scheme, we undertook to evaluate the R.D.G. for space derivatives of the velocity field. As is well known, in turbulence experiments, this quantity is positive and increases strikingly with the order of the derivative[9]. If ϕ_n denotes the R.D.G. of the nth derivative,

$$\phi_n = \frac{\sqrt{\pi}\, A^2 (\nu t)^{-1/2}}{24\,\nu^2} \left(\phi_{n0} + a\,\phi_{n1} + a^2\,\phi_{n2} + b\,\phi_{n3}\right), \tag{12}$$

where ϕ_{nj} are numerical quantities obtained from the $K^{(i)}$ and

$$a = 4\nu B/A^2 - 1 \tag{13}$$

$$b = 1 - 12\nu B/A^2 - 24\nu^2 C/A^3 \tag{14}$$

Computed values of the ϕ_{nj}, $j = 0, 1, 2, 3$ are given in the table below:

n	ϕ_{n0}	ϕ_{n1}	ϕ_{n2}	ϕ_{n3}
0	4.3	4.9	4.5	-.75
1	7.1	9.5	6.4	-2.3
2	14.3	8.9	5.9	-4.5
3	20.6	12.8	6.2	-10.0

In discussing these results we suppose that (1) has been made dimensionless, u being measured in units of $u_o = A\ell^{-3/2}$, x in units of ℓ, t in units of ℓ/u_o; then ν is R^{-1} where $R = u_o\ell/\nu$, the Reynolds number at t = 0. The asymptotic result given in (12) requires times larger than the Reynolds number. Although it is hoped that the representation used here will be useful in the description of flows during times when the Reynolds number is large, we content ourselves here with a regime where the Reynolds number is small or of order unity. In such a case ν is large or of order unity and the correction ϕ_n is small if $\nu t >> 1$. Here a and b may take on various values, difficult to determine from the present state of the theory. In general, for reasonable choices of A, B, and C the deviation from Gaussianity increases with increasing order of the derivative. For instance, it is easily seen that there is an increase if C = 0, and $4\nu B/A^2 > 1$. This choice represents the situation where at the initial instant (of course after the flow has decayed as described above) $K^{(3)} = 0$, and $K^{(2)}$ is order unity or small but not zero. This

increase in non-Gaussianity with order of the derivative is observed experimentally for real fluid flow[9] although for far shorter decay times. It is interesting to compare these results with calculations which have been made of the statistical characteristics of solutions of (1) by Moomaw[10]. He finds that the correction to the Gaussian value for the flatness factor is proportional to $t^{-1/2}$ also; however, the dependence on the Reynolds number is different. In the main, these results are encouraging and suggest that the stochastic representation used here may be useful in the treatment of turbulence problems.

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